DOMINATION SPACES AND FACTORIZATION OF LINEAR AND MULTILINEAR SUMMING OPERATORS

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Abstract. It is well known that not every summability property for multilinear operators leads to a factorization theorem. In this paper we undertake a detailed study of factorization schemes for summing linear and nonlinear operators. Our aim is to integrate under the same theory a wide family of classes of mappings for which a Pietsch type factorization theorem holds. Our construction includes the cases of absolutely p-summing linear operators, (p,σ)-absolutely continuous linear operators, factorable strongly p-summing multilinear operators, $(p_1,\ldots,p_n)$-dominated multilinear operators and dominated $(p_1,\ldots,p_n;\sigma)$-continuous multilinear operators.

1. Introduction and preliminaries

Domination and factorization properties of linear and non linear operators are usually restricted to expressions that involve the p-homogeneous scalar map $t \rightsquigarrow |t|^p$ and its inverse. Actually, even the most abstract approaches to the problem are based in this kind of expressions (see [2, 16, 17]). In [2] the R–S-abstract p-summing mappings were introduced with the aim of unifying the wide variety of summing classes of mappings that can be found in the literature. These abstract approaches provide domination results that recover all the known related dominations for classes of summing mappings in the settings of linear and multilinear operators, homogeneous polynomials, Lipschitz mappings and subhomogeneous mappings among others. However, the main achievements to find abstract domination theorems inspired in absolutely summing operators are not accompanied by general factorization theorems. It is known that although for some classes of multilinear mappings a domination theorem holds, it is sometimes not clear whether this can be written as a standard factorization (see [8, 15]). Some progress was made in [15, 21], where the tandem domination/factorization for classes of summing polynomials and multilinear operators was orchestrated. In this paper we will show a meaningful set of vector norm inequalities that provide not only domination theorems, *but also* the corresponding Pietsch factorization schemes. To do so, we introduce a general family of summability properties for linear and multilinear mappings, that are characterized by means of domination inequalities that *always* lead to factorization theorems. The related class of summing

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operators is defined by means of a homogeneous mapping Φ, and can be seen as a subclass of R–S-abstract p-summing mappings in the sense of [2], that includes the classes of absolutely p-summing linear operators, (p,σ)-absolutely continuous linear operators, factorable strongly p-summing multilinear operators, $(p_1,\ldots,p_n)$-dominated multilinear operators and dominated $(p_1,\ldots,p_n;\sigma)$-continuous multilinear operators. Therefore, our aim is to study these domination properties related to Φ, and to establish the truth of the relations

$$\textit{Summability inequality} \Leftrightarrow \textit{Operator domination} \Leftrightarrow \textit{Operator factorization}$$

for all these classes.

This paper is organized as follows. After this introduction and some preliminaries, in Section 2 we present the factorization theorem through a domination space for the class of linear operators which are defined by a summability property involving a homogeneous map: the Φ-abstract p-summing operators. We use techniques inspired in the convexification of a domination inequality. In the third section we find the factorization of two categories of multilinear maps by using suitable domination spaces: the strongly Φ-abstract p-summing and the $(\Phi_1,...,\Phi_m)$-abstract $(p_1,...,p_m)$-summing multilinear operators. Finally, in the last section we study a multilinear version of (p,σ)-absolutely continuous linear operators in order to apply the technique of factorization previously developed.

Let $m\in\mathbb{N}$ and $X_1,...,X_m,Y$ be Banach spaces over $\mathbb{K}$, (either $\mathbb{R}$ or $\mathbb{C}$). We will denote by $\mathcal{L}\left(X_1,...,X_m;Y\right)$ the Banach space of all continuous m-linear mappings from $X_1\times...\times X_m$ to Y, under the norm

$$\|T\|=\sup_{x^j\in B_{X_j}}\left\|T(x^1,...,x^m)\right\|,$$

where B_{X_j} denotes the closed unit ball of X_j $(j=1,...,m)$. If $Y=\mathbb{K}$, we write $\mathcal{L}\left(X_1,...,X_m\right)$. In the case $X_1=...=X_m=X$, we will simply write $\mathcal{L}\left({}^mX;Y\right)$, whereas $\mathcal{L}(X,Y)$ is the usual Banach space of all continuous linear operators from X to Y.

By $X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m$ we denote the projective tensor product of $X_1,\ldots,X_m$. The projective norm is defined by

$$\pi(v)=\inf\left\{\sum_{i=1}^n\prod_{j=1}^m\left\|x_i^j\right\|,n\in\mathbb{N},v=\sum_{i=1}^n x_i^1\otimes\cdots\otimes x_i^m\right\}.$$

We consider the canonical continuous multilinear mapping

$$\sigma_m:X_1\times\ldots\times X_m\longrightarrow X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m$$

defined by

$$\sigma_m(x^1,...,x^m)=x^1\otimes\cdots\otimes x^m$$

for all $(x^1,\ldots,x^m)\in X_1\times\cdots\times X_m$. Given $T\in\mathcal{L}(X_1,\ldots,X_m;Y)$, consider its linearization $T_L:X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m\to Y$, given by $T_L(x^1\otimes\cdots\otimes x^m)=T\left(x^1,\ldots,x^m\right)$ and extended by linearity. It is well known that $T=T_L\circ\sigma_m$ and the Banach space $\mathcal{L}(X_1,\ldots,X_m;Y)$ is isometrically isomorphic to $\mathcal{L}(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m,Y)$ through the correspondence $T_L\longleftrightarrow T$. For the general theory of tensor products we refer to [6, 22].

Let $1 \leq p < \infty$. Let $\ell_p(X)$ be the Banach space of all absolutely p-summable sequences $(x_i)_{i=1}^{\infty}$ in the Banach space X with the norm

$$\|(x_i)_{i=1}^{\infty}\|_p = (\sum_{i=1}^{\infty} \|x_i\|^p)^{\frac{1}{p}}.$$

We denote by $\ell_{p,\omega}(X)$ the Banach space of all weakly p-summable sequences $(x_i)_{i=1}^{\infty}$ in X with the norm

$$\|(x_i)_{i=1}^{\infty}\|_{p,\omega} = \sup_{x^* \in B_{X^*}} \|(\langle x_i, x^* \rangle)_{i=1}^{\infty}\|_p .$$

Note that $\ell_{p,\omega}(X) = \ell_p(X)$ for some $1 \leq p < \infty$ if, and only if, X is finite dimensional.

Let X, Y and E be (arbitrary) sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Let

$$R\colon K \times E \times G \longrightarrow [0,\infty) \text{ and } S\colon \mathcal{H} \times E \times G \longrightarrow [0,\infty)$$

be arbitrary mappings and $0 < p < \infty$. A mapping $f \in \mathcal{H}$ is said to be *R-S-abstract p-summing* is there is a constant $C > 0$ so that

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^p \right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^{m} R(\varphi, x_j, b_j)^p \right)^{\frac{1}{p}}, \tag{1}$$

for all $x_1, \ldots, x_m \in E$, $b_1, \ldots, b_m \in G$ and $m \in \mathbb{N}$.

Throughout all the paper we will say that a function Φ between Banach spaces X and Y is *homogeneous* if for every positive real number λ, we have that for every $x \in X$, $\Phi(\lambda x) = \lambda \Phi(x)$. This is sometimes called to be positive homogeneous. We will say that it is *bounded* if there is a constant K such that for every $x \in X$, $\|\Phi(x)\| \leq K\|x\|$.

To finish this introductory section, let us provide the definition and main properties on Banach function spaces on finite measure spaces and their p-th powers. Our main references are Chapter 2 (in particular Section 2.2) in [14] and [10, pp.28,51]. Let (Ω, Σ, μ) be a complete finite measure space. A real Banach space $Z(\mu)$ of (equivalence classes of) μ-measurable functions is a *Banach function space* over μ if $Z(\mu) \subset L^1(\mu)$, it contains all the simple functions and, if $\|\cdot\|_{Z(\mu)}$ is the norm of the space, $g \in Z(\mu)$ and f is a measurable function such that $|f| \leq |g|$ μ–a.e., then $f \in Z(\mu)$ and $\|f\|_{Z(\mu)} \leq \|g\|_{Z(\mu)}$ ([10, Def.1.b.17, p.28]). We always have that $L^\infty(\mu) \subset Z(\mu) \subset L^1(\mu)$.

Let $1 \leq p < \infty$. The p-th power of $Z(\mu)$ is defined as the set of functions

$$Z(\mu)_{[p]} := \{f \in L^0(\mu) : |f|^{1/p} \in Z(\mu)\}$$

that is a Banach function space over μ with the norm $\|f\|_{Z(\mu)_{[p]}} := \||f|^{1/p}\|^p_{Z(\mu)}$, $f \in Z(\mu)_{[p]}$ whenever if $Z(\mu)$ is p-convex (with p-convexity constant equal to 1). If the measure μ is clearly fixed in the context, we will write Z and $Z_{[p]}$ instead of $Z(\mu)$ and $Z(\mu)_{[p]}$ respectively.

We will use also some properties of the space $C(K)$ of all continuous functions on a compact Hausdorff topological space K. Let S be a subspace of $C(K)$ and let μ be a probability Borel measure on K. Consider a Banach function space Z over μ. We will say that the identification map $S \to Z$ is well-defined if for every $f \in S$, the map that identifies f with the equivalence class $[f] \in Z$ of all functions that are μ-a.e. equal to f,

is well-defined and continuous. When we write $C(B_{X^*})$ for some Banach space X, B_{X^*} is considered as a compact space with the weak* topology.

2. Factorization of Φ-abstract p-summing linear operators

2.1. **Φ-abstract p-summing linear operators.** Recall that each Banach space X can be considered (isometrically) as a subspace $\widehat{X}$ of $C(B_{X^*})$ just by considering the elements x of X as functions $\langle x, \cdot\rangle : X^* \to \mathbb{R}$ defined as $\langle x, \cdot\rangle(x^*) := \langle x, x^*\rangle$, $x^* \in X^*$. Of course, here B_{X^*} is considered as a compact subset with respect to the weak* topology, and so the elements of $\widehat{X}$ are continuous functions. Throughout the paper we will write i for the isometric embedding $i : X \to \widehat{X} \subset C(B_{X^*})$ given by $i(x) = \langle x, \cdot\rangle$, $x \in X$. Let $\Phi : \widehat{X} \to C(B_{X^*})$ be a homogeneous function and let $1 \leq p < \infty$.

Definition 2.1. Let X and Y be Banach spaces. A linear operator $T : X \longrightarrow Y$ is Φ-*abstract p-summing* if there is a constant $C > 0$ such that for each $x_1, ..., x_n \in X$ and all $n \in \mathbb{N}$, we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p\right)^{1/p} \leq C \left\| \left(\sum_{i=1}^{n} |\Phi(\langle x_i, \cdot\rangle)|^p\right)^{\frac{1}{p}} \right\|_{C(B_{X^*})}.$$

Notice that if Φ is the inclusion map $\widehat{X} \hookrightarrow C(B_{X^*})$, the definition gives the space of absolutely p-summing operators. Let us develop in what follows some relevant examples.

(1) Matter [13] introduced the concept of (p, σ)-absolutely continuous linear operators $(1 \leq p < \infty, 0 \leq \sigma < 1)$, that was developed later by López Molina and the fourth author [11] and extended in [1, 23]. A linear operator $T : X \to Y$ is (p, σ)-*absolutely continuous* if there is a positive constant C such that for all $n \in \mathbb{N}$ and $(x_i)_{i=1}^n \subset X$, we have

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \leq C \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} \left(|\langle x_i, x^*\rangle|^{1-\sigma} \|x_i\|^{\sigma}\right)^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}}. \tag{2}$$

(p, σ)-Absolutely continuous operators are Φ-abstract p-summing when the function Φ is defined by $\Phi(\langle x, \cdot\rangle) := \|x\|^{\sigma} |\langle x, \cdot\rangle|^{1-\sigma}$, $x \in X$. In particular, absolutely p-summing operators are recovered when $\sigma = 0$.

(2) Consider the function $\Phi : X \longrightarrow C(B_{X^*})$ given by $\Phi(\langle x, \cdot\rangle) := \frac{|\langle x, \cdot\rangle|^2}{\|x\|}$, $x \in X$. Take also $p = 1$ and a linear operator $T : X \longrightarrow Y$. The summability inequality can be written as

$$\sum_{i=1}^{n} \|T(x_i)\| \leq C \left\| \sum_{i=1}^{n} \frac{|\langle x_i, \cdot\rangle|^2}{\|x_i\|} \right\|_{C(B_{X^*})}$$

for each $x_1, ..., x_n \in X$. Note that an operator satisfying such a domination property is always 1-summing, since $\frac{|\langle x, \cdot\rangle|^2}{\|x\|} \leq |\langle x, \cdot\rangle|$ for all $x \in X$.

(3) Take $p = 1$. Fix a norm one functional $x_0^* \in X^*$ and consider the function

$$\Phi(\langle x, \cdot\rangle) := |\langle x, x_0^*\rangle|^{1/2} |\langle x, \cdot\rangle|^{1/2}.$$

Note that, if T satisfies such a domination, we obtain that

$$\sum_{i=1}^n \|T(x_i)\| \leq C \left\| \sum_{i=1}^n |\langle x_i, x_0^*\rangle|^{\frac{1}{2}} |\langle x_i, \cdot\rangle|^{\frac{1}{2}} \right\|_{C(B_{X^*})}.$$

Applying [16, Theorem 2.1] to this class of mappings, (see the explanation at the end of this subsection) we find a regular Borel probability measure η on B_{X^*} (with the weak* topology) such that

$$\|T(x)\| \leq C |\langle x, x_0^*\rangle|^{1/2} \int_{B_{X^*}} |\langle x, \cdot\rangle|^{1/2} \, d\eta \leq \frac{1}{2} C \big(|\langle x, x_0^*\rangle| + \int_{B_{X^*}} |\langle x, \cdot\rangle| \, d\eta \big)$$
$$= C \int_{B_{X^*}} |\langle x, \cdot\rangle| \, d\big(\frac{\delta_{x_0^*} + \eta}{2} \big).$$

where $\delta_{x_0^*}$ is the Dirac's delta associated with $x_0^* \in B_{X^*}$. In particular, this shows that this class is contained in the class on absolutely summing operators.

The Φ-abstract p-summing operators form a subclass of R–S-abstract p-summing mappings. To see this, just consider Banach spaces X and Y, $E = X$, $\mathcal{H} = \mathcal{L}(X, Y)$, $G = \mathbb{R}$, $K = B_{X^*}$ endowed with the weak* topology, and define

$$R : B_{X^*} \times X \times \mathbb{R} \to [0, \infty) \quad \text{and} \quad S : \mathcal{L}(X, Y) \times X \times \mathbb{R} \to [0, \infty)$$

by $R(\phi, x, b) := |\Phi(\langle x, \phi\rangle)||b|$ and $S(T, x, b) := \|T(x)\||b|$.

An application of [2, Theorem 2.2] gives that an operator $T : X \to Y$ is Φ-abstract p-summing if and only if there exists a constant $C > 0$ and a regular Borel probability measure μ on B_{X^*} such that

$$\|T(x)\| \leq C \left(\int_{B_{X^*}} |\Phi(\langle x, x^*\rangle)|^p d\mu(x^*) \right)^{1/p}$$

for all $x \in X$.

2.2. **Domination spaces.** Let $\Phi : \widehat{X} \to C(B_{X^*})$ be a homogeneous function. Let Z be a Banach space and suppose that there is a (norm one) continuous linear operator $j : span\{\Phi(\widehat{X})\} \to Z$. We define on $\widehat{X}$ a seminorm associated to Φ and Z as

$$\|\langle x, \cdot\rangle\|_{\Phi, Z} := \inf \sum_{i=1}^r \|j \circ \Phi(\langle x_i, \cdot\rangle)\|_Z,$$

where the infimum is computed over all decompositions of x as $x = \sum_{i=1}^r x_i$, $x, x_i \in X$. In what follows we will assume always that this seminorm is continuous with respect to the norm of $\widehat{X}$.

Consider the subspace $N := \{\langle x, \cdot\rangle \in \widehat{X} : \|\langle x, \cdot\rangle\|_{\Phi, Z} = 0\} \subseteq \widehat{X}$. The completion of the corresponding quotient space $\widehat{X}/N$ is what we call the *domination space* $\widehat{X}_\Phi^Z$.

Let us consider now the quotient map $i_{\widehat{X}_\Phi^Z} : \widehat{X} \to \widehat{X}_\Phi^Z$, that is defined as $i_{\widehat{X}_\Phi^Z}(\langle x, \cdot\rangle) := [\langle x, \cdot\rangle]$, $\langle x, \cdot\rangle \in \widehat{X}$, where $[\langle x, \cdot\rangle]$ denotes the equivalence class of the continuous function

$\langle x, \cdot \rangle$ in $\widehat{X}/N$. Under the assumption that Φ is bounded, the map $i_{\widehat{X}_\Phi^Z}$ is continuous whenever we consider $\widehat{X}$ endowed with the supremum norm induced by $C(B_{X^*})$. Indeed, if $x \in X$ then

$$\begin{aligned}
\|i_{\widehat{X}_\Phi^Z}(\langle x, \cdot \rangle)\|_{\widehat{X}_\Phi^Z} &= \|[\langle x, \cdot \rangle]\|_{\widehat{X}_\Phi^Z} = \|\langle x, \cdot \rangle\|_{\Phi, Z} \\
&\leq \|j \circ \Phi(\langle x, \cdot \rangle)\|_Z \leq \|\Phi(\langle x, \cdot \rangle)\|_{C(B_{X^*})} \\
&\leq K\|\langle x, \cdot \rangle\|_{C(B_{X^*})},
\end{aligned}$$

where K is the constant that comes from the boundedness of Φ.

Let $T : X \to Y$ be a continuous linear operator. Having in mind the classical Pietsch factorization scheme for p-summing operators, the construction of the space $\widehat{X}_\Phi^Z$ aims to define a factorization scheme for T associated to Z and Φ as the one that follows, provided that some suitable inequality holds. Indeed, we will show that the diagram

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "i"'] & Y \\
\widehat{X} \arrow[r, "i_{\widehat{X}_\Phi^Z}"] & \widehat{X}_\Phi^Z \arrow[u, "\widehat{T}"']
\end{tikzcd}$$

is commutative whenever the operator T satisfies an adequate inequality that assures the continuity of $\widehat{T}$.

The main example of this kind of factorization is the classical diagram that holds for p-summing operators. The classical Pietsch domination theorem asserts that an operator $T : X \to Y$ is absolutely p-summing if, and only if, there exists a constant $C > 0$ such that

$$\|T(x)\| \leq C \left(\int_{B_{X^*}} |\langle x, x^* \rangle|^p d\mu(x^*) \right)^{1/p}$$

for all $x \in X$. This domination by the L^p-norm allows to achieve the well-known factorization theorem for p-summing operators through a subspace of $L^p(\mu)$ for the Pietsch measure μ. In general, it is expected that the existence of a domination by a norm of a certain space provides a factorization theorem. Our aim is to show that this is the case for Φ-abstract p-summing operators. In the multilinear case —as it will be shown in the next section— a domination like this does not lead necessarily to a factorization of the multilinear map.

Let us provide other examples of summing inequalities that in fact lead to p-summing type factorization schemes. Following the standard definition, the operator $i_{\widehat{X}_\Phi^Z} : \widehat{X} \to \widehat{X}_\Phi^Z$ is p-concave if there is $C > 0$ such that for $\langle x_1, \cdot \rangle, ..., \langle x_n, \cdot \rangle \in \widehat{X}$,

$$\left(\sum_{i=1}^n \|\langle x_i, \cdot \rangle\|_{\widehat{X}_\Phi^Z}^p \right)^{1/p} \leq C \left\| \left(\sum_{i=1}^n |\langle x_i, \cdot \rangle|^p \right)^{\frac{1}{p}} \right\|_{C(B_{X^*})}.$$

The following result provides a characterization of p-summing operators using our tools.

Proposition 2.2. *Let $T : X \to Y$ be an operator and let $1 \leq p < \infty$. The following statements are equivalent.*

(1) *There is a homogeneous map $\Phi : \widehat{X} \to C(B_{X^*})$, a Banach lattice Z such that $i_{\widehat{X}^Z_\Phi}$ is p-concave and a continuous linear operator $\widehat{T} : \widehat{X}^Z_\Phi \to Y$ such that $T = \widehat{T} \circ i_{\widehat{X}^Z_\Phi} \circ i$.*

(2) *T is p-summing.*

Proof. To prove (1) $\Rightarrow$ (2), note that p-concavity of $i_{\widehat{X}^Z_\Phi}$ implies that it is p-summing. For the converse, just notice that the Pietsch factorization of the p-summing operator gives (1) for $Z = L^p(\eta)$ for a certain Borel probability measure η and Φ just the identity. □

This easy characterization opens the door to easy sufficient conditions that come from our abstract factorization diagram for T to be p-summing. Let T be an operator that factors through $\widehat{X}^Z_\Phi$ as above.

- If $\widehat{X}^Z_\Phi$ is p-concave then T is p-summing, since $i_{\widehat{X}^Z_\Phi}$ is a positive operator (see [10]).
- If Z is p-concave and $|\Phi(\langle x, \cdot \rangle)| \leq C|\langle x, \cdot \rangle|$ for all $x \in X$ and a constant $C > 0$, then T is p-summing.
- If Z is p-concave and $\|(\sum_{i=1}^n |\Phi(\langle x_i, \cdot \rangle)|^p)\|^{1/p} \leq C \|(\sum_{i=1}^n |\langle x_i, \cdot \rangle|^p)^{1/p}\|$ for each finite family $x_1, ..., x_n \in X$, then T is p-summing.

The next result gives the domination characterization of our family of summing operators.

Proposition 2.3. *Let $T : X \to Y$ be an operator between Banach spaces and let $\Phi : \hat{X} \to C(B_{X^*})$ be a homogeneous map. Let be*

(i) *either $Z = C(B_{X^*})$,*

(ii) *or $Z = Z(\mu)$, a p-convex Banach function space over μ —where μ is a Borel probability measure on B_{X^*}— such that the inclusion map $C(B_{X^*}) \to Z$ is well-defined.*

The following assertions are equivalent.

(1) *For each finite set $x_1, ..., x_n \in X$,*

$$\left(\sum_{i=1}^n \|T(x_i)\|^p \right)^{1/p} \leq \left\| \left(\sum_{i=1}^n |\Phi(\langle x_i, \cdot \rangle)|^p \right)^{1/p} \right\|_Z .$$

(2) *There is a positive functional z_p^* in the topological dual of the p-th power of Z such that for each $x \in X$,*

$$\|T(x)\| \leq \left\langle |\Phi(\langle x, \cdot \rangle)|^p, z_p^* \right\rangle^{1/p}.$$

Proof. To prove (1) $\Rightarrow$ (2) use the fact that each p-convex Banach function space has the property that its p-th power $Z_{[p]}$ is (maybe after renorming) a Banach function space. If $Z = C(B_{X^*})$, then simply notice that the p-th power of Z coincides with Z. We may assume w.l.o.g. that the p-convexity constant of Z is 1, and so the quasi-norm given in

the definition of the p-th power is a norm. Furthermore,

$$\left\| \left(\sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p \right)^{1/p} \right\|_Z = \left\| \sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p \right\|_{Z_{[p]}}^{1/p},$$

for each finite set $x_1, \dots, x_n \in X$, and then the inequality in (1) can be written as

$$\sum_{i=1}^{n} \|T(x_i)\|^p - \sup_{g \in B_{(Z_{[p]})^*}} \left\langle \sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p, g \right\rangle \le 0.$$

Consider all functions given by

$$\psi(g) := \sum_{i=1}^{n} \|T(x_i)\|^p - \left\langle \sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p, g \right\rangle,$$

$g \in B_{(Z_{[p]})^*}$. A standard argument using the homogeneity of Φ and Ky Fan's Lemma (see, e.g. [14, Lemma 6.12]) for the concave family of all these convex w*-continuous functions gives the result. The converse is proved by a direct computation. $\square$

Note that for $0 \le z_p^* \in (Z_{[p]})^*$, the expression $f \mapsto \left\langle |f|^p, z_p^* \right\rangle^{1/p}$, $f \in C(B_{X^*})$ defines an abstract p-seminorm, that can be used together with Φ to define an "L^p-type" domination space. The next result shows that in fact this is the canonical domination space, and any other domination defined by another Banach function space Z leads to a factorization through an L^p-type domination space. We show in the next result that the requirement on the existence of a continuous identification map $C(B_{X^*}) \to Z$ always implies such a factorization.

Theorem 2.4. *Let $T : X \to Y$ be an operator between Banach spaces and let Φ a bounded homogeneous map $\Phi : \widehat{X} \to C(B_{X^*})$. The following assertions are equivalent.*

(1) *There is a constant $K > 0$ and a Banach function space Z over a finite Borel measure μ on B_{X^*} such that the identification map $C(B_{X^*}) \to Z$ is well-defined (and so continuous), and for each finite set $x_1, ..., x_n \in X$,*

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{1/p} \le K \left\| \left(\sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p \right)^{1/p} \right\|_Z.$$

(2) *The operator T is Φ-abstract p-summing, that is, there is a constant $K > 0$ such that for each finite set $x_1, ..., x_n \in X$,*

$$\left(\sum_{i=1}^{n} \|T(x_i)\|^p \right)^{1/p} \le K \left\| \left(\sum_{i=1}^{n} |\Phi(\langle x_i, \cdot \rangle)|^p \right)^{1/p} \right\|_{C(B_{X^*})}.$$

(3) *There is a constant $K > 0$ and a Borel probability measure η on B_{X^*} such that for each $x \in X$,*

$$\|T(x)\| \le K \Big(\int_{B_{X^*}} |\Phi(\langle x, \cdot \rangle)|^p \, d\eta \Big)^{1/p}, \quad x \in X.$$

(4) *There is a domination space $(\widehat{X})_\Phi^{L^p}$ defined by Φ and a space $L^p(\eta)$ —for a probability Borel measure η on B_{X^*}—, and a continuous linear operator $\widehat{T}:(\widehat{X})_\Phi^{L^p}\to Y$ such that $T=\widehat{T}\circ i_{(\widehat{X})_\Phi^{L^p}}\circ i$, i.e. the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "T"] \arrow[d, "i"'] & Y \\ \widehat{X} \arrow[r, "i_{(\widehat{X})_\Phi^{L^p}}"] & (\widehat{X})_\Phi^{L^p} \arrow[u, "\widehat{T}"'] \end{tikzcd} .$$

Proof. We assume without loss of generality that $K=1$. The implication (1) to (2) is obvious. That (2) implies (3) is a consequence of [2, Theorem 2.2] or Proposition 2.3, taking into account that the dual of $C(B_{X^*})$ is the space of regular Borel measures on B_{X^*}. For the implication (3) to (4), note that we can define a domination space by means of the seminorm

$$\|\langle x,\cdot\rangle\|_{\Phi,L^p}:=\inf\Big\{\sum_{i=1}^n\Big(\int_{B_{X^*}}|\Phi(\langle x_i,\cdot\rangle)|^p\,d\eta\Big)^{1/p}:\sum_{i=1}^n x_i=x\Big\},\quad x\in X.$$

Define $\widehat{T}([\langle x,\cdot\rangle]):=T(x)$. Let us see that $\widehat{T}$ is well-defined. If $[\langle x,\cdot\rangle]=[\langle y,\cdot\rangle]$ then $\|\langle x-y,\cdot\rangle\|_{\Phi,L^p}=0$. Given $\epsilon>0$ consider a finite decomposition $x-y=\sum_{i=1}^r z_i$ such that $\sum_{i=1}^r\|j\circ\Phi(\langle z_i,\cdot\rangle)\|_{L^p}<\epsilon$. By (3),

$$\|T(x-y)\|\le\sum_{i=1}^r\|T(z_i)\|\le\sum_{i=1}^r\Big(\int_{B_{X^*}}|\Phi(\langle z_i,\cdot\rangle)|^p\,d\eta\Big)^{1/p}<\epsilon,$$

and so $T(x)=T(y)$.

For each x and each finite set $x_1,...,x_n\in X$ such that $\sum_{i=1}^n x_i=x$, we have

$$\|T(x)\|\le\sum_{i=1}^n\|T(x_i)\|\le\sum_{i=1}^n\Big(\int_{B_{X^*}}|\Phi(\langle x_i,\cdot\rangle)|^p\,d\eta\Big)^{1/p},$$

and then $\|T(x)\|\le\|\langle x,\cdot\rangle\|_{\Phi,L^p}$. Hence,

$$\begin{aligned}\|\widehat{T}([\langle x,\cdot\rangle])\| &= \inf\{\|T(y)\|:[\langle x,\cdot\rangle]=[\langle y,\cdot\rangle]\}\\ &\le \inf\{\|\langle y,\cdot\rangle\|_{\Phi,L^p}:[\langle x,\cdot\rangle]=[\langle y,\cdot\rangle]\}\\ &= \|[\langle x,\cdot\rangle]\|_{(\widehat{X})_\Phi^{L^p}}.\end{aligned}$$

By the comments at the beginning of this subsection, if Φ is bounded then the map $i_{(\widehat{X})_\Phi^{L^p}}:\widehat{X}\to(\widehat{X})_\Phi^{L^p}$ is continuous whenever $\widehat{X}$ is endowed with the supremum norm induced by $C(B_{X^*})$. This gives the factorization in (4).

The proof of the implication (4) to (1) follows from the direct calculation:

$$\begin{aligned}\Big(\sum_{i=1}^{n}\|i_{(\widehat{X})^{L^p}_{\Phi}}(\langle x_i,\cdot\rangle)\|^p_{(\widehat{X})^{L^p}_{\Phi}}\Big)^{1/p} &\leq \Big(\sum_{i=1}^{n}\|\langle x_i,\cdot\rangle\|^p_{\Phi,L^p}\Big)^{1/p}\\ &\leq \Big(\sum_{i=1}^{n}\int_{B_{X^*}}|\Phi(\langle x_i,\cdot\rangle)|^p\,d\eta\Big)^{1/p}\\ &\leq \left\|\left(\sum_{i=1}^{n}|\Phi(\langle x_i,\cdot\rangle)|^p\right)^{\frac{1}{p}}\right\|_{C(B_{X^*})},\end{aligned}$$

which shows that $i_{(\widehat{X})^{L^p}_{\Phi}}$ is Φ-abstract p-summing.

□

Let us finish the section with two particular cases of Theorem 2.4.

(1) In the case of p-summing operators, we know that the bounded homogeneous map Φ is the embedding map $\iota:\widehat{X}\to C(B_{X^*})$. In this case, the domination space is just a closed subspace of $L_p(\mu)$.
(2) The domination space for (p,σ)-absolutely continuous operators, is the interpolation space $L_{p,\sigma}(\mu)$ (see for example [5]). Recall that the bounded homogeneous map Φ is given by $\Phi(\langle x,\cdot\rangle):=\|x\|^{\sigma}|\langle x,\cdot\rangle|^{1-\sigma}$.

3. Domination spaces for the multilinear case

3.1. **Strongly Φ-abstract p-summing multilinear operators.** Let $X_1,\cdots,X_m$ be Banach spaces. Consider the compact space $B_{\mathcal{L}(X_1,\dots,X_m)}$, endowed with the weak* topology taking into account that $(X_1\otimes_\pi\cdots\otimes_\pi X_m)^*=\mathcal{L}(X_1,\dots,X_m)$. We will denote by $i_m:X_1\times\dots\times X_m\to C(B_{(X_1\otimes_\pi\cdots\otimes_\pi X_m)^*})$ the m-linear mapping defined as

$$i_m(x^1,\dots,x^m)(\varphi):=\left\langle x^1\otimes\dots\otimes x^m,\varphi\right\rangle,$$

for every $x^j\in X_j$, $j=1,\dots,m$, and $\varphi\in B_{(X_1\otimes_\pi\cdots\otimes_\pi X_m)^*}$. Let V_m be the linear space spanned by the set $i_m(X_1\times\dots\times X_m)$. The elements of this space are the (weak*) continuous functions

$$\sum_{k=1}^{n}\lambda^k\left\langle x^{1,k}\otimes\dots\otimes x^{m,k},\cdot\right\rangle,$$

where $\lambda^k\in\mathbb{R}$ and $x^{j,k}\in X_j$, $j=1,\dots,m$, $k=1,\dots,n$.

Let $\Phi:V_m\to C(B_{(X_1\otimes_\pi\cdots\otimes_\pi X_m)^*})$ be a homogeneous mapping and let $1\leq p<\infty$.

Definition 3.1. A multilinear operator $T:X_1\times\cdots\times X_m\to Y$ is *strongly Φ-abstract p-summing* if there is a constant $C>0$ such that for each $x_i^{j,k}\in X_j$, $(j=1,\dots,m)$, and scalars λ_i^k, $i=1,\dots,n_1$, $k=1,\dots,n_2$ and all $n_1,n_2\in\mathbb{N}$, we have

$$\left(\sum_{i=1}^{n_1}\left\|\sum_{k=1}^{n_2}\lambda_i^kT(x_i^{1,k},\dots,x_i^{m,k})\right\|^p\right)^{1/p}$$

$$\leq C \left\| \left(\sum_{i=1}^{n_1} \left| \Phi \left(\sum_{k=1}^{n_2} \lambda_i^k \left\langle x_i^{1,k} \otimes ... \otimes x_i^{m,k}, \cdot \right\rangle \right) \right|^p \right)^{\frac{1}{p}} \right\|_{C(B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*})} . \tag{3}$$

Note that the class of factorable strongly p-summing multilinear operators introduced in [15] is an example of this definition just defining Φ as the natural embedding map $\iota : V_m \to C(B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*})$. The strongly p-summing multilinear operators of Dimant (see [8]) are also related with our class, but in this case only $n_2 = 1$ is allowed in the inequalities considered in the definition.

The next result gives the characterization of strongly Φ-abstract p-summing m-linear mappings by an integral domination.

Theorem 3.2. *Let $T : X_1 \times ... \times X_m \to Y$ be a m-linear mapping among Banach spaces and let Φ be a homogeneous map $\Phi : V_m \to C(B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*})$. The mapping T is strongly Φ-abstract p-summing if and only if there is a regular Borel probability measure η on $B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*}$ and a constant $C > 0$ such that for each $x^{j,k} \in X$, $\lambda^k \in \mathbb{R}$ with $j = 1, ..., m$ and $k = 1, ..., n$ we have*

$$\left\| \sum_{k=1}^{n} \lambda^k T(x^{1,k}, ..., x^{m,k}) \right\|$$

$$\leq C \left(\int_{B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*}} \left| \Phi \left(\sum_{k=1}^{n} \lambda^k \left\langle x^{1,k} \otimes ... \otimes x^{m,k}, \cdot \right\rangle \right) \right|^p d\eta \right)^{\frac{1}{p}} .$$

Proof. We write $Z^{(\mathbb{N})}$ for all sequences in a Banach space Z that are eventually null. The m-linear mapping T is R-S-abstract p-summing for

$$R : B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*} \times (\mathbb{R} \times X_1, \dots, X_m)^{(\mathbb{N})} \times \mathbb{R} \to [0, \infty)$$

and

$$S : \mathcal{L}(X_1, \dots, X_m; Y) \times (\mathbb{R} \times X_1, \dots, X_m)^{(\mathbb{N})} \times \mathbb{R} \to [0, \infty),$$

given by

$$R\Big(\phi, \big((\lambda^1, x^{1,1}, \dots, x^{m,1}), \dots, (\lambda^n, x^{1,n}, \dots, x^{m,n})\big), b\Big) := \Big|\Phi(\sum_{k=1}^{n} \lambda^k \langle x^{1,k} \otimes \cdots \otimes x^{m,n}, \cdot \rangle)(\phi)\Big|$$

and

$$S\Big(f, \big((\lambda^1, x^{1,1}, \dots, x^{m,1}), \dots, (\lambda^n, x^{1,n}, \dots, x^{m,n})\big), b\Big) := \Big\| \sum_{k=1}^{n} \lambda^k f(x^{1,k}, \dots, x^{m,n}) \Big\|.$$

Then [2] or [16] gives the result.

□

Note that Theorem 3.2 can be proved using [2, Theorem 2.2] (see also the proof of Theorem 3.6 below.)

Now we give the main result of this section, that generalizes Theorem 3.3 in [15].

Theorem 3.3. *Let $1 \leq p < \infty$. Let $T : X_1 \times ... \times X_m \to Y$ be an m-linear mapping between Banach spaces and let Φ a bounded homogeneous map $\Phi : V_m \to C(B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*})$. Then the mapping T is strongly Φ-abstract p-summing if and only if there is a domination space $(\widehat{V_m})_\Phi^{L^p}$ defined by Φ and a L^p-space, and a continuous linear operator $\widehat{T} : (\widehat{V_m})_\Phi^{L^p} \to Y$ such that $T = \widehat{T} \circ i_{(\widehat{V_m})_\Phi^{L^p}} \circ i_m$, i.e.*

$$\begin{tikzcd}
X_1 \times ... \times X_m \arrow[r, "T"] \arrow[d, "i_m"'] & Y. \\
V_m \arrow[r, "i_{(\widehat{V_m})_\Phi^{L^p}}"] & (\widehat{V_m})_\Phi^{L^p} \arrow[u, "\widehat{T}"']
\end{tikzcd}$$

Proof. By Theorem 3.2, if T is strongly Φ-abstract p-summing there is a regular probability measure η on $B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*}$ and a constant $C > 0$ such that for every $x^{j,k} \in X$, $\lambda^k \in \mathbb{R}$ with $j = 1, ..., m$ and $k = 1, ..., n$

$$\left\| \sum_{k=1}^n \lambda^k T(x^{1,k}, ..., x^{m,k}) \right\|$$

$$\leq C \left(\int_{B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*}} \left| \Phi \left(\sum_{k=1}^n \lambda^k \left\langle x^{1,k} \otimes ... \otimes x^{m,k}, \cdot \right\rangle \right) \right|^p d\eta \right)^{\frac{1}{p}}.$$

Therefore, in this case we consider the Banach function space $Z = L^p(\eta)$ and its related domination space $(\widehat{V_m})_\Phi^{L^p}$, whose norm is given by

$$\left\| \left[\sum_{k=1}^n \lambda^k \left\langle x^{1,k} \otimes ... \otimes x^{m,k}, \cdot \right\rangle \right] \right\|_{(\widehat{V_m})_\Phi^{L^p}}$$

$$:= \inf \sum_{i=1}^r \left(\int_{B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*}} \left| \Phi \left(\sum_{k=1}^n \lambda_i^k \left\langle x_i^{1,k} \otimes ... \otimes x_i^{m,k}, \cdot \right\rangle \right) \right|^p d\eta \right)^{\frac{1}{p}},$$

where the infimum is taken over all representations of $\sum_{k=1}^n \lambda^k x^{1,k} \otimes ... \otimes x^{m,k}$ of the form $\sum_{i=1}^r \left(\sum_{k=1}^n \lambda_i^k x_i^{1,k} \otimes ... \otimes x_i^{m,k} \right)$. □

The following corollary gives the connection between a strongly Φ-abstract p-summing multilinear operators and its linearization.

Corollary 3.4. *Let $\Phi : V_m \to C(B_{(X_1 \otimes_\pi \cdots \otimes_\pi X_m)^*})$ be homogeneous and let $1 \leq p < \infty$. A multilinear mapping $T : X_1 \times ... \times X_m \to Y$ is strongly Φ-abstract p-summing if and only if its linearization T_L is a Φ-abstract p-summing linear operator.*

Proof. Note that, by the canonical factorization of the multilinear maps through the projective tensor product, we have that the factorization in Theorem 3.3 can be written as

$$\begin{tikzcd}
& X_1 \times ... \times X_m \arrow[r, "T"] \arrow[dl, "\sigma_m"'] \arrow[d, "i_m"'] & Y \\
X_1 \otimes_\pi \cdots \otimes_\pi X_m \arrow[r, "I_m"] & V_m \arrow[r, "i_{(\widehat{V_m})^{L^p}_\Phi}"] & (\widehat{V_m})^{L^p}_\Phi \arrow[u, "\widehat{T}"']
\end{tikzcd}$$

where I_m is the the isometric embedding $X_1 \otimes_\pi \cdots \otimes_\pi X_m \longrightarrow V_m$ given by

$$I_m(x^1 \otimes \cdots \otimes x^m) = \left\langle x^1 \otimes \cdots \otimes x^m, \cdot \right\rangle$$

It is clear that $\widehat{T} \circ i_{(\widehat{V_m})^{L^p}_\Phi} \circ I_m = T_L$ and so by Theorem 2.4 T is strongly Φ-abstract p-summing if and only if T_L is Φ-abstract p-summing. Note that, using the notation introduced in Section 2, we have that $V_m = (\widehat{X_1 \otimes_\pi \cdots \otimes_\pi X_m})$. □

3.2. $(\Phi_1, ..., \Phi_m)$**-abstract** $(p_1, ..., p_m)$**-summing multilinear operators.**

Throughout this section, $1 \leq p, p_1, ..., p_m < \infty$ are such that $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$ and $X_1, ..., X_m, Y$ are Banach spaces. Let also $\Phi_j : \widehat{X}_j \longrightarrow C(B_{X_j^*})$, $j = 1, ..., m$, be bounded homogeneous maps.

Definition 3.5. An m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be $(\Phi_1, ..., \Phi_m)$-abstract $(p_1, ..., p_m)$-summing if there is a constant $C > 0$ so that

$$\left(\sum_{i=1}^{n} \left\| T(x_i^1, ..., x_i^m) \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} \left\| \left(\sum_{i=1}^{n} \left| \Phi_j \left(\left\langle x_i^j, . \right\rangle \right) \right|^{p_j} \right)^{\frac{1}{p_j}} \right\|_{C(B_{X_j^*})}.$$

Using the full general Pietsch Domination Theorem presented by Pellegrino et all in [17] we obtain the Domination Theorem for our class.

Theorem 3.6. *An operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is $(\Phi_1, ..., \Phi_m)$-abstract $(p_1, ..., p_m)$-summing if and only if there is a constant $C > 0$ and regular Borel probability measures $\mu_j \in C(B_{X_j^*})^*$ such that for all $x^j \in X_j (j = 1, ..., m)$,*

$$\left\| T(x^1, ..., x^m) \right\| \leq C \prod_{j=1}^{m} \left(\int_{B_{X_j^*}} \left| \Phi_j \left(\langle x^j, . \rangle \right) \right|^{p_j} d\mu_j \right)^{\frac{1}{p_j}}. \tag{4}$$

Proof. The m-linear mapping T is $R_1, ..., R_m$-S-abstract $(p_1, ..., p_m)$-summing (see [17, Definition 4.4]), for

$$R_j(\varphi, b^1, ..., b^r, x^j) = \left| \Phi_j \left(\left\langle x^j, \varphi \right\rangle \right) \right|$$

and

$$S(T, b^1, ..., b^r, x^1, ..., x^m) = \left\| T(x^1, ..., x^m) \right\|$$

for all $\varphi \in B_{X_j^*}, b^l \in \mathbb{K}, x^j \in X_j$ with $j = 1, ..., m$. Then Theorem 4.6 in [17] gives the result. □

Using the factorization for the linear case (Theorem 2.4), we are now ready to construct the domination space that provides the factorization theorem for our class.

Theorem 3.7. *The m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is $(\Phi_1, ..., \Phi_m)$-abstract $(p_1, ..., p_m)$-summing if and only if there are Banach spaces G_j and Φ_j-abstract p_j-summing linear operators $u_j : X_j \longrightarrow G_j$, j=1,...m, such that T factors through $G_1 \times ... \times G_m$ as $T = \widehat{T} \circ (u_1, ..., u_m)$, where $\widehat{T} \in \mathcal{L}(G_1, ..., G_m; Y)$.*

Proof. A simple computation using Theorem 2.4 and Theorem 3.6 shows that, if T has such a factorization then T is $(\Phi_1, ..., \Phi_m)$-abstract $(p_1, ..., p_m)$-summing.

Conversely, take T in our class. For each $j = 1, ...m$, consider the map $u_j : X_j \to (\widehat{X_j})_{\Phi_j}^{L^{p_j}}$ defined by $u_j := i_{(\widehat{X_j})_{\Phi_j}^{L^{p_j}}} \circ i_j$, where $i_j : X_j \to \widehat{X}_j$ is the identification map and $i_{(\widehat{X_j})_{\Phi_j}^{L^{p_j}}} : \widehat{X}_j \to (\widehat{X_j})_{\Phi_j}^{L^{p_j}}$ is the quotient map defined as was explained in Section 2. Notice that we have

$$\begin{aligned} \left\| u_j(x^j) \right\| &= \inf \left\{ \sum_{k=1}^{n} \left(\int_{B_{X_j^*}} \left| \Phi_j \left(\left\langle x_k^j, . \right\rangle \right) \right|^{p_j} d\mu_j \right)^{\frac{1}{p_j}}, x^j = \sum_{k=1}^{n} x_k^j \right\} \\ &\leq \left(\int_{B_{X_j^*}} \left| \Phi_j \left(\left\langle x^j, . \right\rangle \right) \right|^{p_j} d\mu_j \right)^{\frac{1}{p_j}}, \end{aligned}$$

and so u_j is Φ_j-abstract p_j-summing for all $j = 1, ...m$. Let $\widehat{T_0}$ be the m-linear mapping defined on $u_1(X_1) \times ... \times u_m(X_m)$ by

$$\widehat{T_0}(u_1(x^1) \times ... \times u_m(x^m)) := T(x^1, ..., x^m),$$

that is continuous due to the inequality (4) that provides Theorem 3.6. (Following the idea of [4, Theorem 3.6] we can easily prove that the mapping $\widehat{T_0}$ is well-defined and continuous). We finish the proof by defining $\widehat{T}$ as the unique extension of $\widehat{T_0}$ to $\overline{u_1(X_1)} \times ... \times \overline{u_m(X_m)} = (\widehat{X_1})_{\Phi_1}^{L^{p_1}} \times ... \times (\widehat{X_m})_{\Phi_m}^{L^{p_m}}$. □

Example 3.8. The situation considered in the preceding theorem includes relevant well-known cases. The next classes of operators satisfy factorization theorems associated to summability properties that are particular cases of Theorem 3.7.

(1) $(p_1, ..., p_m)$**-dominated multilinear operators** (see [20] or [12]). Let $1 \leq p, p_1, ..., p_m < \infty$ with $\frac{1}{p} = \frac{1}{p_1} + ... + \frac{1}{p_m}$. A multilinear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is $(p_1, ..., p_m)$-*dominated* if there is a constant $C > 0$ such that

$$\left(\sum_{i=1}^{n} \left\| T(x_i^1, ..., x_i^m \right\|^p \right)^{\frac{1}{p}} \leq C \prod_{j=1}^{m} \sup_{x_j^* \in B_{X_j^*}} \left(\sum_{i=1}^{n} \left| \left\langle x_i^j, x_j^* \right\rangle \right|^{p_j} \right)^{\frac{1}{p_j}}$$

for all $(x_i^j)_{1 \leq i \leq n} \subset X_j$, $j = 1, ..., m$. In this case, the corresponding homogeneous maps Φ_j are the inclusion maps $\widehat{X}_j \hookrightarrow C(B_{X_j^*})$ and the corresponding seminorms

are

$$\left\|\left\langle x^j,.\right\rangle\right\|_{(\widehat{X_j})^{L^{p_j}}_{\Phi_j}} = \inf\left\{\sum_{k=1}^{n}\left(\int_{B_{X_j^*}}\left|\left\langle x_k^j,.\right\rangle\right|^{p_j}d\eta_j\right)^{\frac{1}{p_j}}, x^j=\sum_{k=1}^{n}x_k^j\right\}.$$

The domination spaces $(\widehat{X_j})^{L^{p_j}}_{\Phi_j}$ coincide with a subspace of $L^{p_j}(\eta_j)$, $j=1,...,m$, that provide the factorization theorem for this class (see Theorem 3.2.4 in [9] and Theorem 14 in [20]).

(2) **Dominated** $(p_1,...,p_m;\sigma)$**-continuous multilinear operators** (see [4] or [5]). Let $1\le p,p_1,...,p_m<\infty$ and $0\le\sigma<1$ with $\frac{1}{p}=\frac{1}{p_1}+...+\frac{1}{p_m}$. A multilinear operator $T\in\mathcal{L}(X_1,...,X_m;Y)$ is *dominated* $(p_1,...,p_m;\sigma)$*-continuous* if there is a constant $C>0$ such that

$$\left(\sum_{i=1}^{n}\left\|T(x_i^1,...,x_i^m)\right\|^{\frac{p}{1-\sigma}}\right)^{\frac{1-\sigma}{p}} \le C\prod_{j=1}^{m}\sup_{x_j^*\in B_{X_j^*}}\left(\sum_{i=1}^{n}\left(\left|\left\langle x_i^j,x_j^*\right\rangle\right|^{1-\sigma}\left\|x_i^j\right\|^{\sigma}\right)^{\frac{p_j}{1-\sigma}}\right)^{\frac{1-\sigma}{p_j}}$$

for all $(x_i^j)_{1\le i\le n}\subset X_j$, $j=1,...,m$. The corresponding homogeneous maps $\Phi_j:\widehat{X}_j\longrightarrow C(B_{X_j^*})$ are defined by

$$\Phi_j(\left\langle x^j,.\right\rangle)=\left|\left\langle x^j,.\right\rangle\right|^{1-\sigma}\left\|x^j\right\|^{\sigma},$$

the corresponding seminorms are given by

$$\left\|\left\langle x^j,.\right\rangle\right\|_{(\widehat{X_j})^{L^{p_j}}_{\Phi_j}} = \inf\left\{\sum_{k=1}^{n}\left\|x_k^j\right\|^{\sigma}\left(\int_{B_{X_j^*}}\left|\left\langle x_k^j,.\right\rangle\right|^{p_j}d\eta_j\right)^{\frac{1-\sigma}{p_j}}, x^j=\sum_{k=1}^{n}x_k^j\right\}$$

and the domination spaces $(\widehat{X_j})^{L^{p_j}}_{\Phi_j}$ coincide with the spaces $L_{p_j,\sigma}(\eta_j)$, $j=1,...,m$, that were described in [5, Section 2.1.3]. Then Theorem 3.7 when applied with these elements gives the factorization result that can be found in [5, Theorem 2.1.20] (see also [1] and [4]).

4. Applications: the particular case of factorable strongly summing multilinear operators

Dimant introduced in [8] the ideal of strongly p-summing multilinear operators as follows.
A continuous m-linear operator $T:X_1\times...\times X_m\longrightarrow Y$ is *strongly* p*-summing* if there exists a constant $C>0$ such that for every $x_1^j,...,x_n^j\in X_j$, $j=1,...,m$, we have

$$\left\|\left(T(x_i^1,...,x_i^m)\right)_{i=1}^{n}\right\|_p \le C\sup_{\varphi\in B_{\mathcal{L}(X_1,...,X_m)}}\left(\sum_{i=1}^{n}\left|\varphi(x_i^1,...,x_i^m)\right|^p\right)^{\frac{1}{p}}. \qquad (5)$$

In this section we will follow the notation of the original presentation of Dimant according to the above definition. We will denote by $\mathcal{L}_p^s(X_1,...,X_m;Y)$ the vector space of all strongly p-summing m-linear operators T from $X_1\times...\times X_m$ into Y, which is a Banach

space if we consider the norm $\|T\|_{\mathcal{L}_p^s}$, the infimum of all C satisfying (5). Even if this class satisfies a domination theorem, it is well-known that it does not satisfy a factorization theorem. So, no factorization scheme is expected whenever we try to extend this class via an interpolation procedure. Matter initiated this general interpolation technique in [13] where he introduces the class of (p,σ)-absolutely continuous linear operators, that recovers the p-summing operators for $\sigma = 0$. The natural interpolation class related to strongly p-summing m-linear maps can be constructed as follows.

Let $1 \leq p < \infty$ and $0 \leq \sigma < 1$. For all $(x_i^j)_{i=1}^n \subset X_j, (1 \leq j \leq m)$ we put

$$\delta_{p\sigma}((x_i^j)_{i=1}^n) = \sup_{\varphi \in B_{\mathcal{L}(X_1,...,X_m)}} \left(\sum_{i=1}^n \left(\left|\varphi(x_i^1,...,x_i^m)\right|^{1-\sigma} \prod_{j=1}^m \left\|x_i^j\right\|^{\sigma} \right)^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

It is clear that

$$\sup_{\varphi \in B_{\mathcal{L}(X_1,...,X_m)}} \left(\sum_{i=1}^n \left|\varphi(x_i^1,...,x_i^m)\right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}} \leq \delta_{p\sigma}((x_i^j)_{i=1}^n),$$

for all $(x_i^j)_{i=1}^n \subset X_j$, $1 \leq j \leq m$.

Definition 4.1. For $1 \leq p < \infty$ and $0 \leq \sigma < 1$, a mapping $T \in \mathcal{L}(X_1,...,X_m;Y)$ is *Dimant strongly (p,σ)-continuous* if there is a constant $C > 0$ such that for any $x_1^j,...,x_n^j \in X_j$, $1 \leq j \leq m$, we have

$$\left\|\left(T\left(x_i^1,...,x_i^m\right)\right)_{i=1}^n\right\|_{\frac{p}{1-\sigma}} \leq C\ \delta_{p\sigma}((x_i^j)_{i=1}^n). \tag{6}$$

The class of all Dimant strongly (p,σ)-continuous m-linear operators from $X_1 \times ... \times X_m$ into Y, which is denoted by $\mathcal{L}_p^{s,\sigma}(X_1,...,X_m;Y)$ is a Banach space with the norm $\|T\|_{\mathcal{L}_p^{s,\sigma}}$ which is the smallest constant C such that the inequality (6) holds.
For $\sigma = 0$, we have $\mathcal{L}_p^{s,0}(X_1,...,X_m;Y) = \mathcal{L}_p^{s}(X_1,...,X_m;Y)$. Actually this vector space has to be thought of as an intermediate space between the space of all strongly p-summing mappings ($\sigma = 0$) and the whole class of continuous m-linear mappings ($\sigma = 1$).

Theorem 4.2. *Let $1 \leq p \leq \infty$ and $0 \leq \sigma < 1$. A m-linear operator $T \in \mathcal{L}(X_1,...,X_m;Y)$ is Dimant strongly (p,σ)-continuous if and only if there is a constant $C > 0$ and regular Borel probability measure μ on $B_{\mathcal{L}(X_1,...,X_m)}$ so that for all $\left(x^1,...,x^m\right) \in X_1 \times .... \times X_m$ the inequality*

$$\left\|T(x^1,...,x^m)\right\| \leq C \left(\int_{B_{\mathcal{L}(X_1,...,X_m)}} (\left|\phi(x^1,...,x^m)\right|^{1-\sigma} \prod_{j=1}^m \left\|x^j\right\|^{\sigma})^{\frac{p}{1-\sigma}} d\mu(\phi) \right)^{\frac{1-\sigma}{p}}, \tag{7}$$

is valid. Moreover, we have in this case

$$\|T\|_{\mathcal{L}_p^{s,\sigma}} = \inf\{C > 0 : \textit{for all } C \textit{ satisfying the inequality (7)}\}$$

Proof. For the proof we use the abstract version of Pietsch's domination theorem presented by Botelho et al in [2]. Note that by choosing the parameters

$$\begin{cases} X = E = X_1 \times ... \times X_m \\ \mathcal{H} = \mathcal{L}\left({}^m X_1, ..., X_m; Y\right) \\ x_o = (0, ..., 0) \\ K = B_{\mathcal{L}(X_1,...,X_m)} \\ G = \mathbb{K} \\ R(\varphi, x^1, ..., x^m, \lambda) = |\lambda| \left|\varphi(x^1, ..., x^m)\right|^{1-\sigma} \prod_{j=1}^{m} \left\|x_i^j\right\|^{\sigma} \\ S(T, (x^1, ..., x^m), \lambda) = |\lambda| \left\|T(x^1, ..., x^m)\right\|. \end{cases}$$

We can easily conclude that $T : X_1 \times ... \times X_m \longrightarrow Y$ is Dimant strongly (p, σ)-continuous if and only if T is R-S- abstract $\frac{p}{1-\sigma}$-summing (see [2, Definition 2.1]), and by [2, Theorem 2.2] we recover (7). □

An immediate consequence of Theorem 4.2 is the following corollary.

Corollary 4.3. *Consider* $0 \le \sigma < 1$ *and* $1 \le p, q < \infty$ *with* $p \le q$. *If* $T \in \mathcal{L}_p^{s,\sigma}\left(X_1, ..., X_m; Y\right)$ *then* $T \in \mathcal{L}_q^{s,\sigma}\left(X_1, ..., X_m; Y\right)$ *and* $\|T\|_{\mathcal{L}_q^{s,\sigma}} \le \|T\|_{\mathcal{L}_p^{s,\sigma}}$

Proposition 4.4. *Let* $1 \le p < \infty, 0 \le \sigma < 1$. *Then*

$$\mathcal{L}^s_{\frac{p}{1-\sigma}}\left(X_1, ..., X_m; Y\right) \subset \mathcal{L}_p^{s,\sigma}\left(X_1, ..., X_m; Y\right).$$

By [3, Example 3.3] there is a strongly p-summing bilinear operator which is not weakly compact. Then the previous proposition proves the existence of a Dimant strongly (p, σ)-continuous bilinear operator which is not weakly compact.

The lack of factorization scheme for the class of Dimant strongly (p, σ)-continuous operators highlights the need of approaching the definition to a factorable context. A precedent was done first in [21], where a general condition related to the existence of a factorization scheme was isolated. Combining the ideas in [21] with those in [8], the class of factorable strongly p-summing operators was introduced in [15], that best inherits the spirit of absolutely summing linear operators. Following this scheme, we now combine Dimant strongly (p, σ)-continuous operators with the ideas in [21] to produce a class for which a factorization theorem occurs.

Definition 4.5. For $1 \le p < \infty$ and $0 \le \sigma < 1$, a mapping $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is factorable strongly (p, σ)-continuous if there is a constant $C > 0$ such that for any natural numbers n_1, n_2 and vectors $x_{i,k}^j \in X_j$, $1 \le j \le m$, and scalars λ_i^k, $i = 1, ..., n_1$, $k = 1, ..., n_2$ we have

$$\left(\sum_{i=1}^{n_1} \left\| \sum_{k=1}^{n_2} \lambda_i^k T(x_i^{1,k}, ..., x_i^{m,k}) \right\|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}$$

$$\le C. \sup_{\varphi \in B_{\mathcal{L}(X_1 \times \cdots \times X_n)}} \left(\sum_{i=1}^{n_1} \left| \sum_{k=1}^{n_2} |\lambda_i^k| |\varphi(x_i^{1,k}, ..., x_i^{m,k})|^{1-\sigma} \prod_{j=1}^{m} \|x_i^{j,k}\|^{\sigma} \right|^{\frac{p}{1-\sigma}} \right)^{\frac{1-\sigma}{p}}.$$

Remark 4.6. 1) It is clear that this definition is equivalent to saying that T is strongly Φ-abstract $\frac{p}{1-\sigma}$-summing with the homogeneous function Φ defined —using our notation— by

$$\Phi\left(\sum_{k=1}^{n}\lambda_i^k\left\langle x^{1,k}\otimes...\otimes x^{m,k},\cdot\right\rangle\right)=\inf\left\{\sum_{k=1}^{n}|\lambda_i^k|\left|\left\langle x^{1,k}\otimes...\otimes x^{m,k},\cdot\right\rangle\right|^{1-\sigma}\prod_{j=1}^{m}\|x^{j,k}\|^{\sigma}\right\},$$

where the infimum is computed for all possible decompositions of the tensor

$$\sum_{k=1}^{n}\lambda_i^k x^{1,k}\otimes...\otimes x^{m,k}.$$

2) Note that if we take $n_2=1$ and $\lambda_i^1=1$ for all $i=1,...,n_1$ in the Definition 4.5 we find that T is Dimant strongly (p,σ)-continuous m-linear mappings.

From Theorem 3.2 and Theorem 3.3 we obtain the following characterization.

Theorem 4.7. *Let $1\leq p<\infty$, $0\leq\sigma<1$ and $T\in\mathcal{L}(X_1,...,X_m;Y)$. The following assertions are equivalent.*

(1) *T is factorable strongly (p,σ)-continuous.*
(2) *There is a constant $C>0$ and a regular Borel probability measure η on $B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m)^*}$ such that for all $x^{j,k}\in X_j$ and $\lambda^k\in\mathbb{R}$, $j=1,...,m$, $k=1,...,n$,*

$$\left\|\sum_{k=1}^{n}\lambda^k T(x^{1,k},...,x^{m,k})\right\|$$

$$\leq C\left(\int_{B_{\mathcal{L}(X_1,...,X_m)}}\left|\sum_{k=1}^{n}|\lambda^k||\varphi(x^{1,k}\otimes...\otimes x^{m,k})|^{1-\sigma}\prod_{j=1}^{m}\|x^{j,k}\|^{\sigma}\right|^{\frac{p}{1-\sigma}}d\eta\right)^{\frac{1-\sigma}{p}}.$$

(3) *The following diagram commutes*

$$\begin{tikzcd}
X_1\times...\times X_m \arrow[r, "T"] \arrow[d, "i"'] & Y \\
V_m \arrow[r, "i_{p,\sigma}"] & L_{p,\sigma}(\eta) \arrow[u, "\hat{T}"']
\end{tikzcd},$$

where $L_{p,\sigma}(\eta)$ is the domination space defined by the seminorm

$$\left\|\sum_{k=1}^{n}\lambda^k\left\langle x^{1,k}\otimes...\otimes x^{m,k},\cdot\right\rangle\right\|_{p,\sigma}$$

$$=\inf\sum_{i=1}^{r}\left(\int_{B_{(X_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi X_m)^*}}\left|\sum_{k=1}^{m}|\lambda_i^k|\left|\left\langle x_i^{1,k}\otimes...\otimes x_i^{m,k},\cdot\right\rangle\right|^{1-\sigma}\prod_{j=1}^{m}\|x_i^{j,k}\|^{\sigma}\right|^{\frac{p}{1-\sigma}}d\eta\right)^{\frac{1-\sigma}{p}},$$

and the infimum is taken over all representations of the form

$$\sum_{k=1}^{n} \lambda^k x^{1,k} \otimes ... \otimes x^{m,k} = \sum_{i=1}^{r} \sum_{k=1}^{m} \lambda_i^k x_i^{1,k} \otimes ... \otimes x_i^{m,k}.$$

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